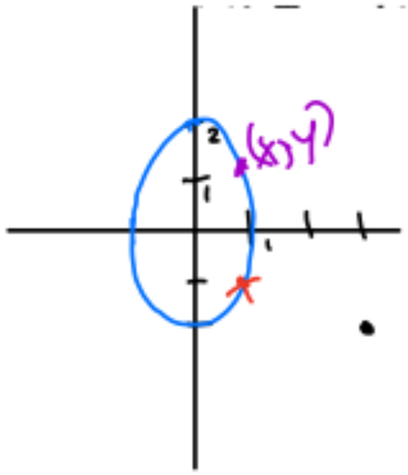
Let's hold on to our papers:

4.47 Find the point on the graph of $4x^2 + y^2 = 4$ that is closest to $(3, -2)$.



We are trying to minimize distance from (x,y) to $(3,-2)$.

$$F = \sqrt{(x-3)^2 + (y+2)^2}$$

$4x^2+y^2=4$ $\quad y^2 = 4-4x^2$

$y = \pm\sqrt{4-4x^2}$ (choose $y = -\sqrt{4-4x^2}$ see picture)

$y = -\sqrt{4(1-x^2)} = -2\sqrt{1-x^2}$

$$\Rightarrow F(x) = \sqrt{(x-3)^2 + (-2\sqrt{1-x^2}+2)^2}$$

Interval: $-1 \le x \le 1$

Pro-tip: minimizing distance at x $\Longleftrightarrow$ minimizing distance^2 at x

(Because $\sqrt{\bullet}$ is an increasing fcn.)

Let's change:

$$\begin{cases} F(x) = (x-3)^2 + (-2\sqrt{1-x^2}+2)^2 & \text{find abs min.} \\ -1 \le x \le 1 \end{cases}$$

$$F'(x) = 2(x-3) + 2(-2\sqrt{1-x^2}+2)\cdot(-2\sqrt{1-x^2}+2)'$$

$$= 2x-6 + \cancel{2}\underbrace{(-2\sqrt{1-x^2}+2)}_{2(-\sqrt{1-x^2}+1)}\cdot(-2)\tfrac{1}{\cancel{2}}(1-x^2)^{-1/2}\cdot(-2x)$$

$$F'(x) = 2x-6+8x(-\sqrt{1-x^2}+1)(1-x^2)^{-1/2}$$

$$\Rightarrow F'(x) = 2x - 6 + 8x\left(-1 + \frac{1}{\sqrt{1-x^2}}\right)$$

$$= 2x - 6 - 8x + \frac{8x}{\sqrt{1-x^2}}$$

$$F'(x) = -6x - 6 + \frac{8x}{\sqrt{1-x^2}} = 0$$

$$\frac{8x}{\sqrt{1-x^2}} = 6x+6 = 6(x+1)$$

$$\frac{1}{\sqrt{1-x^2}} = \frac{6(x+1)}{8x} = \frac{3}{4}\left(1+\frac{1}{x}\right)$$

$$\frac{1}{1-x^2} = \frac{9}{16}\left(1+\frac{1}{x^2}+\frac{2}{x}\right)$$ a little yucky.

$\Longrightarrow x = 0.8042$ is only solution

F ∪ ← must be global min!

$F'(x)$ − − − − 0 + + +

−1 ——— 0.8042 ——— 1

$$y = -2\sqrt{1-x^2}$$

$$= -1.1887$$

closest pt is $\boxed{(0.8042, -1.1887)}$

Just for fun: Let's do it another way:

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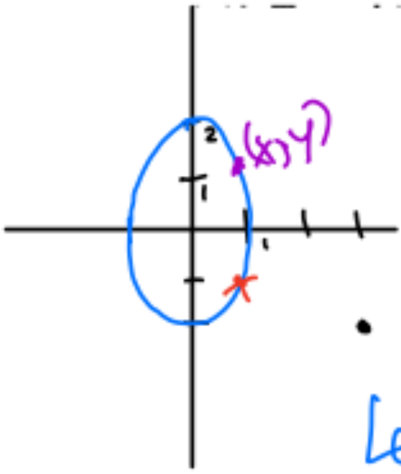

We are trying to minimize distance from (x,y) to $(3,-2)$.

$$F = \sqrt{(x-3)^2 + (y+2)^2}$$

Let's parametrize the ellipse.

$\hookrightarrow$ This is unit circle with y's multiplied by 2.

Any pt: $(x,y) = (\cos\theta, 2\sin\theta)$. $0 \le \theta < 2\pi$

Check: $4x^2+y^2 = 4\cos^2\theta + (2\sin\theta)^2 = 4\cos^2\theta + 4\sin^2\theta$
$= 4(\cos^2\theta + \sin^2\theta) = 4\cdot 1 = 4$ ✓

$\Rightarrow$ Let $F(\theta) = (\cos\theta - 3)^2 + (2\sin\theta + 2)^2$

minimize, $0 \le \theta \le 2\pi$

$$F'(\theta) = 2(\cos\theta - 3)\cdot(-\sin\theta) + 2(2\sin\theta + 2)\cdot(2\cos\theta)$$

$$= -2\sin\theta\cos\theta + 6\sin\theta + 8\sin\theta\cos\theta + 8\cos\theta$$

$$F'(\theta) = 6\sin\theta\cos\theta + 6\sin\theta + 8\cos\theta = 0$$

sagemath $\Rightarrow$ $\theta = \sim 0.6364$

$x = \cos\theta \doteq 0.8042$

$y = 2\sin\theta = -1.1887$.

Type some Sage code below and press Evaluate.

```
f(x)=6*sin(x)*cos(x)+6*sin(x)+8*cos(x)
a=(f(x)).find_root(-pi/2,0,x)
show(a)
show(cos(a))
show(2*sin(a))
```

Evaluate

```
−0.636428161009046
0.8042237227343638
−1.1886533620720232
```

Yet another way to do this:

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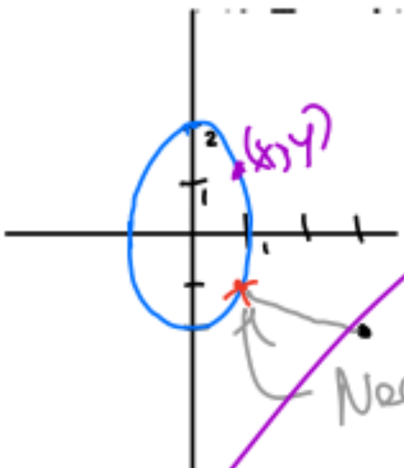

Need to find (x,y) on the graph where slope from (x,y) to $(3,-2)$ is the negative reciprocal of the slope of the tangent line.

slope of tangent: $8x + 2y \cdot \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-8x}{2y} = \frac{-4x}{y}$$

slope of connecting line

$$= \boxed{\frac{y+2}{x-3} = \frac{y}{4x}}$$

Negative reciprocal

solve for x & y.

$$\boxed{4x^2+y^2=4}$$

$$y^2 = 4-4x^2 = 4(1-x^2)$$

$$y = -2\sqrt{1-x^2}$$